Renormalons in Quantum Mechanics

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Outline

- large order perturbation theory and renormalons

- renormalization of 2d QM with δ potential

- renormalons in QM

Perturbation theory

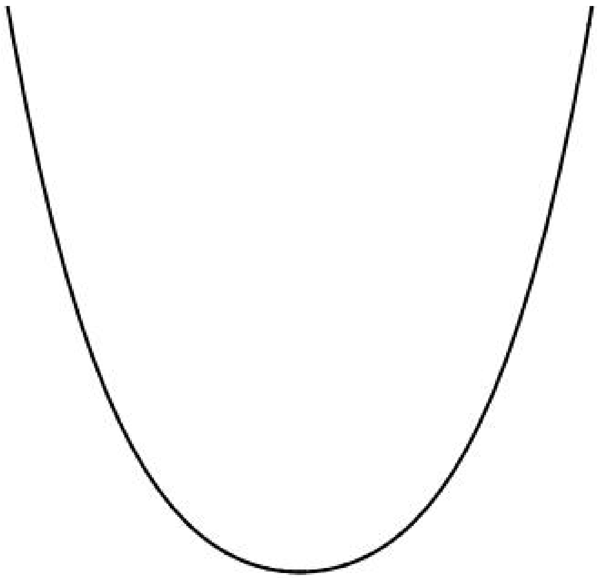
$$\langle \mathcal{O} \rangle \sim \sum_{n=0}^{\infty} a_n \lambda^n$$

Many series expressions in physics are asymptotic series rather than convergent series

Perturbation theory

Observation by Dyson:
If quantity makes no sense at negative coupling the perturbation series can **not** be convergent

Example: $V(\lambda) = x^2 + \lambda x^4$



$\lambda > 0$

$\lambda < 0$

Perturbation theory

$$\langle \mathcal{O} \rangle \sim \sum_{n=0}^{\infty} a_n \lambda^n \qquad a_n \sim A^{-n}(n-k)!$$

Deal with (factorially) divergent series by Borel summation

$$\langle \mathcal{O} \rangle = \sum_{n=0}^{k-1} a_n \lambda^n + \int_0^\infty ds\, e^{-\frac{s}{\lambda}} \sum_{n=k}^{\infty} \frac{a_n}{(n-k)!} s^{n-k}$$

Replaces divergence with term of order $e^{-\frac{A}{\lambda}}$

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convergent series, but pole at $s = A$

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Borel summation is ambiguous when $A > 0$

$$\mathrm{amb}\langle \mathcal{O} \rangle = \mp \pi i \left(\frac{\lambda}{A} \right)^{k-1} e^{-\frac{A}{\lambda}}$$

Non-Perturbative contributions

Asymptotic series with growth

$$a_n \sim A^{-n}(n-k)!$$

completed by order $e^{-\frac{A}{\lambda}}$ contributions

	Origin $n!$ growth	non-perturbative effect
Generically	$\mathcal{O}(n!)$ diagrams of $\mathcal{O}(1)$	instanton(s)

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Renormalon	$\mathcal{O}(1)$ diagrams of $\mathcal{O}(n!)$	unclear

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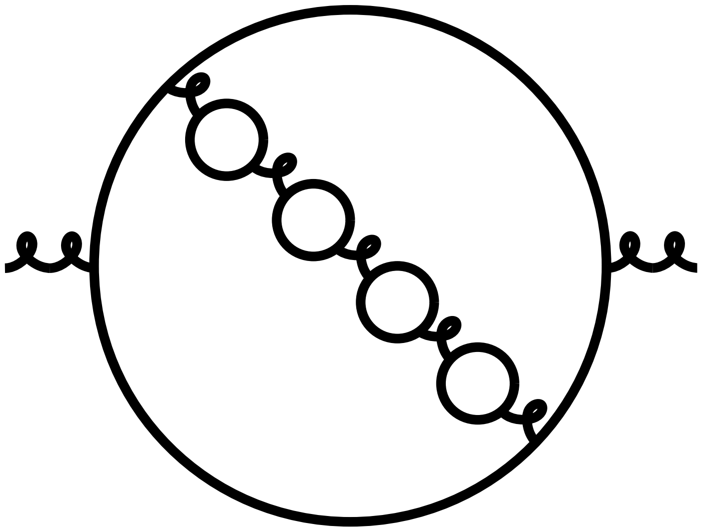
Renormalon	$\mathcal{O}(1)$ diagrams of $\mathcal{O}(n!)$	unclear

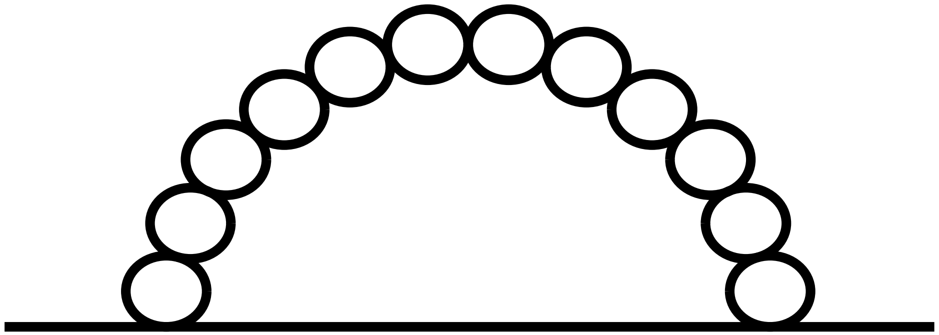
problematic when Borel sum is ambigious
(sometimes OPE can resolve this)

Renormalons

QCD



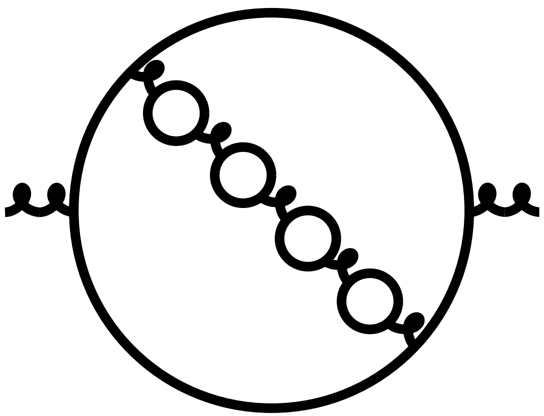
ϕ^4



- renormalized bubble $\sim \log p^2$
- n renormalized bubbles $\sim (\log p^2)^n$
- n renormalized bubbles in a bubble $\sim \int dp^2 f(p^2)(\log p^2)^n$

Renormalons

QCD



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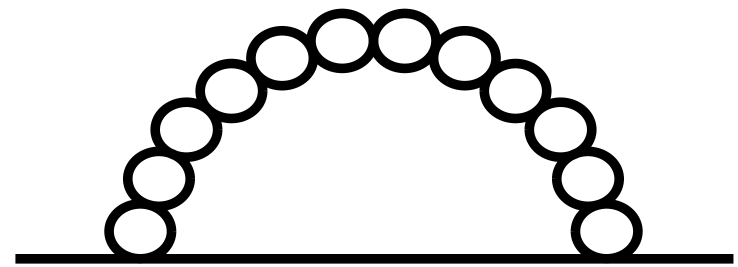

$$\int dp^2 f(p^2)(\log p^2)^n$$

for large n dominated at $\quad p^2 \ll 1 \quad$ and $\quad p^2 \gg 1$

IR	UV
$f(p^2) \sim p^{2k+2}$	$f(p^2) \sim p^{-2l-2}$
$k^n \int^\infty dt e^{-t} t^n$	$l^n \int^\infty dt e^{-t} t^n$

$$\boxed{\sim A^n n!}$$

Renormalons in QM: motivation

Divergences related to 'instantons', examples in

- ODE's
- QM
- QFT

Simple & Rigourous

Complicated & Heuristic

Divergences related to 'renormalons', examples in

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Simple & Rigourous

Known non-perturbative definition

Complicated & Heuristic

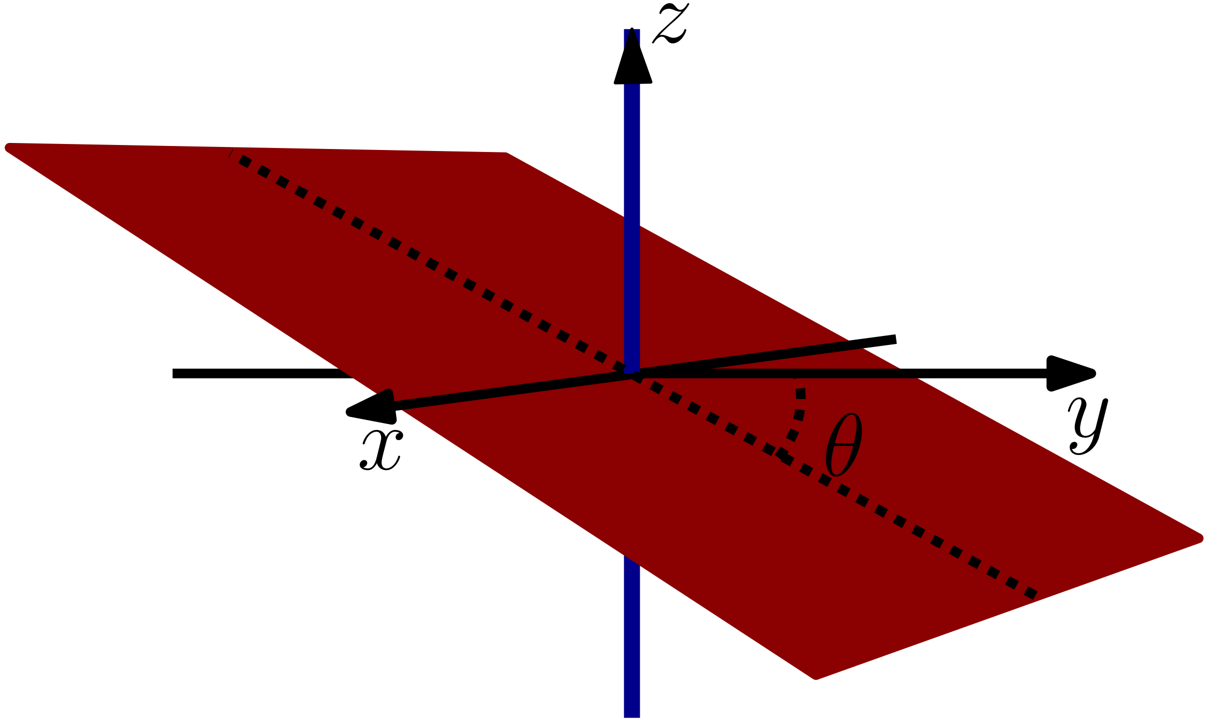
Renormalons in QM

The model:

$$H = p^2 + \lambda_0 V_\star + \kappa V_*$$

$$V_\star = \delta(x)\delta(y)$$

$$V_* = \delta\left(\cos\theta\, z - \sin\theta\, y\right)$$



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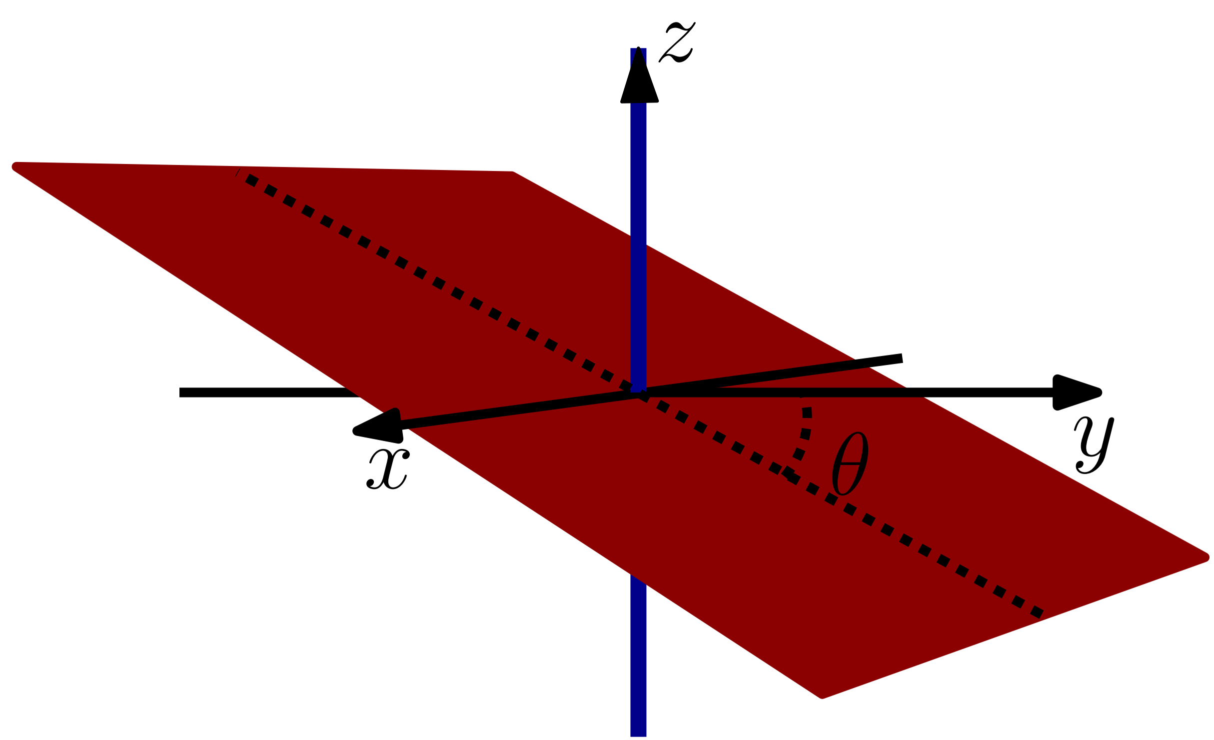

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We show that:

- Perturbative S-matrix has renormalon divergence
- Borel ambiguity is resolved by causality/$i\epsilon$-prescription

$\lim_{\epsilon\to 0}$ and $\lim_{n\to\infty}$ don't commute!

QM of the 2d δ potential

Feynman rules

$$\mathbf{p} = (\mathbf{u}, q) = (v, w, q)$$

$$\star \;:\; \lambda_0 \hat{V}_\star(\mathbf{p}_{k-1} - \mathbf{p}_k) \qquad - \;:\; \int \frac{d^3\mathbf{p}_k}{(2\pi)^3} \frac{1}{p_\mathrm{f}^2 + i\epsilon - p_k^2}$$

1-loop

$$\star - \star \;= 2\pi\delta(q_\mathrm{f} - q_\mathrm{i}) \frac{\lambda_0^2}{4\pi} \int_0^\infty \frac{du^2}{u_\mathrm{f}^2 + i\epsilon - u^2}$$

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1-loop $\quad$ renormalized

$$\star - \star \;=\; 2\pi\delta(q_{\mathrm{f}} - q_{\mathrm{i}})\, \lambda^2\, l(u_{\mathrm{f}})$$

$$l(z) = \frac{1}{4\pi} \log \frac{e^{i\pi} z}{\mu}$$

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(n-1)-loop renormalized

$$\star - \ldots - \star \ = 2\pi\delta(q_\mathrm{f} - q_\mathrm{i})\, \lambda^n\, l(u_\mathrm{f}^2)^{n-1}$$

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all order perturbation theory

$$\boxed{t(\mathbf{p}_\mathrm{f}, \mathbf{p}_\mathrm{i}) = 2\pi\delta(q_\mathrm{f} - q_\mathrm{i}) \frac{\lambda}{1 - \frac{\lambda}{4\pi}(\log \frac{u_\mathrm{f}^2}{\mu} + i\pi)}}$$

QM of the 2d δ potential

$$t(\mathbf{p}_\mathrm{f}, \mathbf{p}_\mathrm{i}) = 2\pi\delta(q_\mathrm{f} - q_\mathrm{i}) \frac{\lambda}{1 - \frac{\lambda}{4\pi}\left(\log\frac{u_\mathrm{f}^2}{\mu} + i\pi\right)}$$

- $$\beta(\lambda) = \frac{\lambda^2}{4\pi}$$
- $$\bar{\lambda}(p^2) = \frac{\lambda}{1 - \frac{\lambda}{4\pi}\log\frac{p^2}{\mu}} = \frac{4\pi}{\log\frac{\Lambda}{p^2}} \qquad \Lambda = \mu e^{\frac{4\pi}{\lambda}}$$
- $$E_\mathrm{b} = -\Lambda$$

renormalized all order perturbative = exact (self-adj. extension)

A renormalon diagram in QM

Add the 1d δ potential: $\quad \ast \; : \; \kappa\, \hat{V}(\mathbf{p}_{k-1}, \mathbf{p}_k)$

All order in λ, second order in κ

$$\ast - \star - \ldots - \star - \ast \;=\; \lambda^n \kappa^2 \cos^2\theta \int \frac{dq}{2\pi} f(q)\, l(p_{\mathrm{f}}^2 - q^2)^{n-1}$$

$$f(q) = \left((q_{\mathrm{f}} - q)(q + \tilde{q}_{\mathrm{f}}) + i\epsilon\right) \left((q_{\mathrm{i}} - q)(q + \tilde{q}_{\mathrm{i}}) + i\epsilon\right)^{-1}$$

A renormalon diagram in QM

Add the 1d δ potential: $\quad * \;:\; \kappa \hat{V}(\mathbf{p}_{k-1}, \mathbf{p}_k)$

All order in λ, second order in κ

$$* - \star - \ldots - \star - * \;\sim\; 2\cos^2\theta\,\kappa^2\,\mu^{-3/2}\left(\frac{\lambda}{6\pi}\right)^n (n-1)!$$

$$\text{ambiguity} \;:\; \pm 2\pi i\,\kappa^2\,\cos^2\theta\,\Lambda^{-3/2}$$

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$$\text{ambiguity} \ : \ \pm 2\pi i\,\kappa^2 \ \cos^2\theta\,\Lambda^{-3/2}$$

Cannot be complete answer: doesn't vanish at $\theta = 0$

A renormalon in QM

Consider all diagrams with two $*$'s

$$* - \star - \ldots - \star - *$$

$$\underbrace{\star - \ldots - \star}_{a} - * - \underbrace{\star - \ldots - \star}_{n-a} - *$$

$$* - \underbrace{\star - \ldots - \star}_{n-a} - * - \underbrace{\star - \ldots - \star}_{a}$$

$$+ \quad \underbrace{\star - \ldots - \star}_{a} - * - \underbrace{\star - \ldots - \star}_{n-a-b} - * - \underbrace{\star - \ldots - \star}_{b}$$

$$t^{(n,2)}(\mathbf{p}_\mathrm{f}, \mathbf{p}_\mathrm{i}) \sim \frac{9}{2}(\cos\theta\ \log\cos^2\theta)^2\kappa^2\mu^{-\frac{3}{2}}\left(\frac{\lambda}{6\pi}\right)^n (n-3)!$$

various cancellations: subleading growth, consistent $\theta \to 0$

Non-perturbative contribution

Borel ambiguity:

$$\mathrm{amb}\, t^{(2)}(\mathbf{p}_{\mathrm{f}}, \mathbf{p}_{\mathrm{i}}) = \pm 2\pi i (\cos\theta \, \log\cos^2\theta)^2 \kappa^2 \Lambda^{\frac{3}{2}} \left(\frac{\lambda}{4\pi}\right)^2$$

Alternative summation

$$\sum_n \int dq\, f(q)\, l(p_{\mathrm{f}}^2 - q^2)^{n-1} \lambda^n \qquad \text{divergent}$$

$$\downarrow$$

$$\int dq\, f(q) \sum_n l(p_{\mathrm{f}}^2 - q^2)^{n-1} \lambda^n$$

$$= \int dq\, f(q) \frac{\lambda}{1 - \lambda\, l(p_{\mathrm{f}}^2 - q^2)} \qquad \text{divergent}$$

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Borel ambiguity:

$$\operatorname{amb} t^{(2)}(\mathbf{p}_\mathrm{f}, \mathbf{p}_\mathrm{i}) = \pm 2\pi i (\cos\theta \, \log\cos^2\theta)^2 \kappa^2 \Lambda^{\frac{3}{2}} \left(\frac{\lambda}{4\pi}\right)^2$$

Alternative summation

$$\sum_n \int dq\, f(q)\, l(p_\mathrm{f}^2 - q^2)^{n-1} \lambda^n \qquad \text{divergent}$$

$$\downarrow$$

$$\int dq\, f(q) \sum_n l(p_\mathrm{f}^2 - q^2)^{n-1} \lambda^n$$

$$= \int dq\, f(q) \frac{\lambda}{1 - \lambda\, l(p_\mathrm{f}^2 - q^2 \pm i\epsilon)} \qquad \text{ambiguous}$$

Non-perturbative contribution

Borel ambiguity resolved:

$$t^{(2)}(\mathbf{p}_\mathrm{f}, \mathbf{p}_\mathrm{i}) = +2\pi i(\cos\theta\, \log\cos^2\theta)^2\kappa^2\Lambda^{\frac{3}{2}}\left(\frac{\lambda}{4\pi}\right)^2 + \dots$$

Alternative summation

$$\sum_n \int dq\, f(q)\, l(p_\mathrm{f}^2 - q^2)^{n-1}\lambda^n$$

$$\downarrow$$

$$\int dq\, f(q) \sum_n l(p_\mathrm{f}^2 - q^2)^{n-1}\lambda^n$$

$$= \int dq\, f(q) \frac{\lambda}{1 - \lambda\, l(p_\mathrm{f}^2 - q^2 + i\epsilon)}$$

causality

Conclusions

- 1-particle nonrelativistic QM can have renormalons
- lead to non-perturbative imaginary contribution

  $\longrightarrow$ checked vs exact solution of the model

Open questions

- independent (semi-classical?) interpretation of non-pert effect
- other (simpler) models?
- other observables?
- general theory?
- useful for QFT?